Lecture 6. (10/6/2021)

From Lecture 5 notes

- Thm 1 (iii) + pf.

Distance function.

For $A \subseteq X$, we define $d(\cdot, A) : X \to \underbrace{}_{[0,\infty)}$ by $d(x,A) = \inf\{d(x,y) : y \in A\}$.

Prop 1. (i) $d(x,A) = d(x,\overline{A})$.

(ii) $d(x,A) = 0 \Leftrightarrow x \in \overline{A}$.

(iii) $|d(x,A) - d(y,A)| \le d(x,y)$

Rem. (iii) $\Rightarrow$ $d(\cdot,A)$ is Lipschitz (see Conway) $\Rightarrow$ $d(\cdot, A)$ is unif. cont.

Pf. (i) Clearly, $d(x,\overline{A}) \le d(x,A)$.

Suppose $d(x,\overline{A}) < d(x,A)$ for some x.

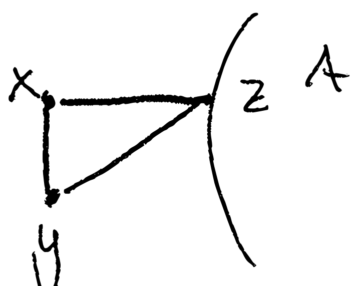
Then, $\exists\, y_0 \in \overline{A}$ s.t. $d(x,y_0) < d(x,A)$. But $y_0 \in \overline{A} \Rightarrow \exists$ seq. $y_n \in A$ s.t. $y_n \to y_0$.

Then, $d(x,A) \leq d(x,y_n) \to d(x,y_0)$, since $d(x,\cdot)$ is continuous. This contradicts $d(x,y_0) < d(x,A)$.

(ii) DIY.

(iii) Pick $\varepsilon > 0$ and $z \in A$ s.t.
$d(x,A) \geq d(x,z) - \varepsilon \quad \Rightarrow$

$$d(y,A) - d(x,A) \leq d(y,z) - (d(x,z) - \varepsilon)$$
$$\leq d(y,x) + \varepsilon$$



Thus, $d(y,A) - d(x,A) \leq d(y,x)$ since $\underbrace{\varepsilon}_{\text{arbitr.}}$.
By symmetry in x,y we get
$$|d(x,A) - d(y,A)| \leq d(x,y). \quad \square$$

Given two sets $A, B \subseteq X$ (nonempty),

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$
$$= \inf\{d(x,B) : x \in A\}.$$

<u>Thm 2.</u> If A compact, B closed, $A \cap B = \emptyset$, then $d(A,B) > 0$.

<u>Pf.</u> Let $f(x) := d(x,B)$. Then, f is cont. by Prop 1 (iii). Since A is compact, by prev. cor., f achieves a minimum on A, i.e. $\exists\, a \in A$ s.t. $f(a) = \inf_{x \in A} f(x) = d(A,B)$.

But $f(a) = d(a,B) = 0 \iff a \in \overline{B} = B$ since B is closed. But $A \cap B = \emptyset$, so $f(a) = d(A,B) > 0$. $\boxtimes$

Uniform convergence.

Def 1 A seq. of fcns $f_n : X \to \Omega$ **converges unif.** to $f : X \to \Omega$ if $\forall \varepsilon > 0 \ \exists N$ s.t. $\rho(f_n(x), f(x)) < \varepsilon \ \forall x \in X$, $n \geq N$.

Thm 3. If $f_n : X \to \Omega$ are contin. and $f_n \to f : X \to \Omega$ unif., then f is continuous.

Pf. Pick $x_0 \in X$, $\varepsilon > 0$. WTS $\exists \delta > 0$ s.t. $\rho(f(x), f(x_0)) < \varepsilon$ if $d(x, x_0) < \delta$.

Pick N s.t. $\rho(f_n(x), f(x)) < \varepsilon/3 \ \forall x$ and $n \geq N$. Fix such n, and choose $\delta > 0$ s.t. $\rho(f_n(x), f_n(x_0)) < \varepsilon/3$ if $d(x, x_0) < \delta$. Then,

$$\rho(f(x), f(x_0)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(x_0)) + \rho(f_n(x_0), f(x_0)) < \varepsilon. \quad \blacksquare$$

Weierstraß Majorant Thm. Suppose $u_n : X \to \mathbb{C}$ are majorized, $|u_n(x)| \leq M_n$ for all $x \in X$ and some seq. M_n. If $\sum_{k=1}^{\infty} M_n < \infty$, then $S_n(x) = \sum_{k=1}^{n} u_k(x)$ converges unif. to some $S : X \to \mathbb{C}$.

Pf. We show $S_n(x)$ is a uniformly Cauchy seq. Since $\mathbb{C}$ is complete, $\exists\, S(x) = \lim_{n\to\infty} S_n(x)$ and the conv. is unif. (DIY). Thus, pick $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} M_k < \infty$ $\exists N$ s.t. $\sum_{k=n}^{m} M_k < \varepsilon$ $\forall m \geq n \geq N$. But then

$$|S_n(x) - S_m(x)| \leq \sum_{k=n}^{m} |u_k(x)| \leq \sum_{n}^{m} M_k < \varepsilon.$$

$\boxtimes$